The Riemann Mapping Theorem

Complex Analysis: From Zeri to Arzela-Ascoli

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Contents

1 Introduction

The purpose of this project is to lay the foundations for a dissertation in complex dynamics: studying the behaviour of iterated complex functions. The author writes for those who, like himself, have only previously studied a "Complex Methods" course — typically delivered to students of the physical sciences and focused on computing integrals using residues without rigorous derivations. Therefore Chapter 2 will lead to several variations of Cauchy's Integral Theorem and Cauchy's Integral Formula: the core driving tools of complex analysis. However, for the sake of brevity (and assuming familiarity with the statement of basic complex analytic theorems, if not their proof) we do not reproduce an entire first course on complex analysis. Results such as the identity theorem/isolated zeros are cited when needed, though they can be proved straightforwardly once the Cauchy Integral Theorem/Cauchy Integral Formula have been established.

The Riemann Mapping Theorem is chosen as a capstone goal: the theorem says that for any simply connected proper subdomain $U \subset \mathbb{C}$, with some base point $z_0 \in U$, there exists a unique bijective holomorphic function f mapping U to the the open unit disc $\mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0)$ is real and positive. Bijective holomorphic functions are known as conformal isomorphisms and in this case we say U is conformally isomorphic to $\mathbb{D}$. This result is powerful as a theorem as it allows us to study the behaviour of complex dynamics on quite unpleasant domains (so long as they are simply connected) by mapping to a disc and studying the behaviour there.

In the proof of the Riemann Mapping Theorem we use a "square root trick" to show that a candidate function is indeed surjective: we consider the set of all functions $\mathcal{F} \coloneqq \{g : U \to \mathbb{D} \mid g \text{ is holomorphic, } g \text{ is injective, } g(z_0) = 0,\ g'(z_0) > 0\}$ and show that there exists some $f \in \mathcal{F}$ with a greater derivative at z_0 than any other function in $\mathcal{F}$. We then show that if f were not surjective then we could use a branch of the square root to find another function in $\mathcal{F}$ with larger derivative at z_0, contradicting f having the greatest derivative. The square root trick can be applied iteratively so that any member of $\mathcal{F}$ will converge to f under repeated iteration — we do not investigate the iteration process in this project but it is an example of behaviour that will be investigated in the dissertation which follows.

The analysis of complex dynamics frequently investigates the topology of sets which are invariant under iterated complex mappings. For this reason, the present text places extra emphasis on the topological theory used to prove the Riemann Mapping Theorem. In particular Chapter 3 develops the topology of complex function spaces, introducing normal families and tests for normality. This entire chapter is devoted to proving that the closure of our set $\mathcal{F}$ is compact, that all members of the closure are also holomorphic and that the map from a function $g \in \mathcal{F}$ to $g'(z_0)$ is continuous with respect to the topology we establish. This is what enables us to prove the existence of some $f \in \mathcal{F}$ with maximal derivative at the base point z_0.

After reaching a minor technical result in Chapter 4, which enables us to prove that the function $f \in \mathcal{F}$ with maximal derivative is in fact injective as claimed, we explicitly combine our results in Chapter 5 and prove the Riemann Mapping Theorem.

This presentation of The Riemann Mapping Theorem closely follows the approach in (Conway, 1995). However the author has added explicit arguments where these are left implicit: for example Corollary 4.3 has been added where Conway refers directly to Hurwitz's Theorem. Individual proofs are a synthesis of (Conway, 1995), (Ahlfors, 1978), (Stein and Shakarchi, 2003) and the author's own preferred exposition.

2 Basic Complex Analysis

The goal of this section is to arrive at the core driving theorems in complex analysis — several versions of Cauchy's Integral Theorem and also the Cauchy Integral Formula. These underpin essentially all later complex analytic results and we give some useful examples in section 2.3.

Additionally, we will prove the existence of logarithms and use this to define branches of the square root function.

For clarity we recall the definition of holomorphicity.

Definition 2.1
A complex function f is holomorphic at a point z_0 if there is some $\epsilon > 0$ such that the complex derivative

$$f'(z) \coloneqq \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists for every point $z \in B(z_0; \epsilon)$.

A complex function f is holomorphic on a set U if it is holomorphic at every point $z_0 \in U$.

2.1 Integrals over paths

Definition 2.2 (Curve)
Let $[a, b]$ be a closed interval of real numbers. We define a curve to be a C^1-smooth map $\gamma\colon [a, b] \to \mathbb{C}$ where C^1-smooth indicates that both the real and imaginary parts of γ have continuous derivatives in the ordinary real sense — $\gamma(t) = \gamma_{\Re}(t) + i\gamma_{\Im}(t)$ with $\gamma_{\Re}$ and $\gamma_{\Im}$ real and $\gamma' := \gamma'_{\Re} + i\gamma'_{\Im}$

However, it is less restrictive to work with *piecewise* C^1-smooth maps.

Definition 2.3 (Path)
We take path to mean a collection of curves $\gamma = \gamma_1, \gamma_2, ..., \gamma_n$ such that the end point of γ_i is the beginning of γ_{i+1} — i.e. if γ_j has domain $[a_j, b_j]$ then $b_i = a_{i+1}$ and $\gamma_i(b_i) = \gamma_{i+1}(a_{i+1})$. Without loss of generality, we assume $a_1 = 0$ and $b_n = 1$ so that γ defines a piecewise C^1-smooth map $\gamma\colon [0, 1] \to \mathbb{C}$

It will sometimes be convenient to refer to a point z in the image of a path γ as being "in" γ — abusing notation and treating γ synonymously with its image. Likewise we will occasionally refer to sets which can be covered with a path as being paths and assume that the appropriate function will be clear.

Definition 2.4 (Complex Integral Over a Path)
Let γ be a path. Let f be a complex function defined on the image of γ. We can split f into its real and imaginary parts $f = f_{\Re} + i f_{\Im}$. If $f_{\Re}$ and $f_{\Im}$ are both Riemann integrable then we define

$$\begin{aligned}\int_\gamma f(z)\,dz &:= \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\,dt \\ &= \int_0^1 f_{\Re}(\gamma(t)) \cdot \gamma'_{\Re}(t) - f_{\Im}(\gamma(t)) \cdot \gamma'_{\Im}(t)\,dt \\ &+ i \int_0^1 f_{\Im}(\gamma(t)) \cdot \gamma'_{\Re}(t) + f_{\Re}(\gamma(t)) \cdot \gamma'_{\Im}(t)\,dt\end{aligned}$$

In practice f will be continuous (except possibly at isolated points) so this guarantees Riemann integrability.

Definition 2.5 (Trigonometric Functions)
For $z \in \mathbb{C}$ we define

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$$

For $z \in \mathbb{C}$ we define

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!}$$

Remark 2.6. We cite (Ahlfors, 1978, pp. 43-45) and claim that the exponential series converges on the whole complex plane and that it is holomorphic. Moreover $\exp' = \exp$. It has all of the usual properties we associate with the real exponential — e.g. $\exp(a+b) = \exp(a) \cdot \exp(b)$. Additionally, $\exp(2\pi i) = 1$ so that $\exp(z + 2\pi i) = \exp(z)$ so the function exp is periodic with minimum period $2\pi i$. We also claim that $\exp(\mathbb{C}) = \mathbb{C}\backslash 0$ so that for any $z_0 \in \mathbb{C}$ we may find (one of many) $z^* \in \mathbb{C}$ such that $\exp(z^*) = z_0$.

We cite (Spivak, 2006, pp.300-311 p.555) and claim that the trigonometric functions sin and cos defined as above exactly match the classical geometric notions of sin and cos when the argument is real.

Proposition 2.7
If $\gamma = e^{2\pi i \cdot t}, 0 \leq t \leq 1$ and $n \in \mathbb{Z}$ then

$$\int_\gamma z^n\,dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Proof. We use the properties of the exp function outlined in Remark 2.6.

$$\begin{aligned}
\int_\gamma z^n \, dz &= \int_0^1 e^{2\pi i \cdot nt} \cdot 2\pi i \cdot e^{2\pi i \cdot t} \, dt \\
&= 2\pi i \cdot \int_0^1 e^{2\pi i \cdot (n+1)t} \, dt \\
&= 2\pi i \cdot \int_0^1 \cos(2\pi i \cdot (n+1)t) \, dt - 2\pi \cdot \int_0^1 \sin(2\pi i \cdot (n+1)t) \, dt \\
&= \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

□

2.2 Cauchy's Theorems

The key result we will need to show that the logarithm is well defined is the guaranteed existence of primitives for holomorphic functions on simply connected domains.

Theorem 2.8 (Primitives exist for holomorphic functions on simply connected domains)
Let $U \subseteq \mathbb{C}$ be a simply connected domain and $f \colon U \to \mathbb{C}$ be holomorphic. Then there exists a holomorphic primitive $F \colon U \to \mathbb{C}$ such that $F' = f$

The proof will follow after a series of lemmas and intermediate theorems...

Definition 2.9 (Domain)
If $U \subseteq \hat{\mathbb{C}}$ is path-connected, non-empty and open then we call it a domain.

Theorem 2.10 (Goursat's Theorem)
Let U be a domain and suppose $f : U \to \mathbb{C}$ is holomorphic. If $T \subset U$ is a triangle and $\partial T \subset U$ is a closed path along its boundary, then

$$\int_{\partial T} f(z) \, dz = 0 \tag{2}$$

Proof. Define $T^0 := T$ to be our intitial triangle (see figure 1). Take l to be the length of ∂_{T^0}. By joining the midpoints of each edge, we may construct four subtriangles T_a^0, T_b^0, T_c^0 and T_d^0 which are each similar to T^0 with each side being half as long (see figure 2) — it does not matter in which order we label the subtriangles. Notice that the internal paths occur twice in opposite directions so

$$\int_{\partial T^0} f(z) \, dz = \sum_{i \in \{a,b,c,d\}} \int_{\partial T_i^0} f(z) \, dz$$

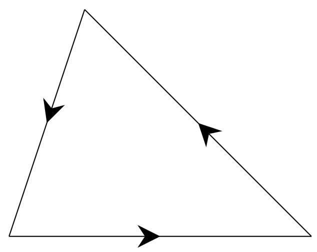


Figure 1: Triangle $T^0 := T$

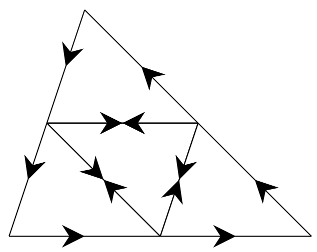


Figure 2: Triangle $T^0 \equiv T_a^0 \cup T_b^0 \cup T_c^0 \cup T_d^0$

Let $I := |\int_{\partial T^0} f(z)\,dz\,|$ then for some $j \in \{a, b, c, d\}$ we must have

$$\left|\int_{\partial T_j^0} f(z)\,dz\right| \geq \frac{I}{4}$$

Define $T^1 := T_j^0$ for this particular j. Since all the subtriangles T_a^0, T_b^0, T_c^0, T_d^0 are similar to T^0 but scaled by a factor of a half then

$$\text{length}(\partial T^1) = \frac{l}{2} \tag{3}$$

We can iterate this subtriangulation process to obtain $T^0 \supseteq T^1 \supseteq T^2 \supseteq \ldots$ such that

$$\left|\int_{\partial T^n} f(z)\,dz\right| \geq \frac{I}{4^n} \tag{4}$$

and

$$\text{length}(\partial T^n) = \frac{l}{2^n} \tag{5}$$

Since the $\{T^n\}_{n=1}^{\infty}$ is a nested sequence of closed sets, it is a topological fact (Munkres, 1999, p. 170) that there is some $z_0 \in \bigcap_{n=1}^{\infty} T^n$. Consider some $\epsilon > 0$. Since f is holomorphic at z_0 then we can find a $\delta > 0$ such that $|z - z_0| < \delta$ implies

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| < \epsilon|z - z_0|$$

Take n such that $T^n \subset B(z_0; \epsilon)$. Note that $z \in T^n$ implies $|z - z_0| < \text{length}(\partial T^n)$. Note also that

$$\int_{\partial T^n} f(z_0) + z_0 \cdot f'(z_0)\,dz = 0$$

and

$$\int_{\partial T^n} z \cdot f'(z_0)\,dz = 0$$

as the functions 1 and z have well defined primitives (namely z and $\frac{1}{2}z^2$) on T^n and z_0, $f(z_0)$ and $f'(z_0)$ are simply constants. Now

$$\begin{aligned}
\left|\int_{\partial T^n} f(z)\,dz\right| &= \left|\int_{\partial T^n} f(z) - f(z_0) - (z - z_0)\cdot f'(z_0)\,dz\right| \\
&\le \int_{\partial T^n} |f(z) - f(z_0) - (z - z_0)\cdot f'(z_0)|\,dz \\
&< \int_{\partial T^n} \epsilon|z - z_0|\,dz \\
&\le \text{length}(\partial T^n)\epsilon \sup_{z\in\partial T^n} |z - z_0| \\
&< \epsilon\ \text{length}(\partial T^n)^2
\end{aligned}$$

Substituting in equations (4) and (5) we have

$$\frac{I}{4^n} \le \frac{l^2}{4^n}\epsilon$$

So

$$I \le l^2\epsilon$$

Since l is fixed and ϵ was arbitrary then $I = 0$. □

We can now use Goursat's theorem to prove a similar theorem, with weaker hypotheses. In particular, we relax the requirement that the function is holomorphic everywhere on the domain. Instead we allow it to be non-holomorphic at a finite number of points, so long as it is still continuous everywhere in the domain.

Theorem 2.11 (Cauchy's Integral Theorem for Triangles)
Let U be a domain and suppose $f : U \to \mathbb{C}$ is continuous on the whole domain and holomorphic except at finitely many points. If $T \subset U$ is a triangle and $\partial T \subset U$ is a closed path along its boundary then

$$\int_{\partial T} f(z)\,dz = 0 \tag{6}$$

Proof. Without loss of generality, suppose that f is holomorphic everywhere except at one point a in the interior of T (non-holomorphicity on ∂T does not affect the integration or our bounds). Since the interior is open then for some $\delta_a > 0$ we have $B(a;\delta_a) \in T$. Since f is continuous at a, then for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $|z - a| < \delta_\epsilon$ implies $|f(z) - f(a)| < \epsilon$. Taking $\delta = \min(\delta_a, \delta_\epsilon)$ we can construct a smaller triangle T^* inscribed in the closed ball $\overline{B(a;\frac{\delta}{2})}$ such that $a \in T^* \subset T$ (see figure 3).
Add new vertices to form a subtriangulation of T (it is an algebraic-topological theorem that this can always be done). Integrate f around each subtriangle *except* T^* (see figure 4) and notice that the interior edges cancel leaving only the edges of T (in the correct orientation) and T^* (in the negative orientation) — see figure 5.
Since f is holomorphic on all of the subtriangles *except* T^* then by Theorem 2.10 (Goursat) these integrals are all zero and so is their sum. Therefore

$$0 = \int_{\partial T} f(z)\,dz - \int_{\partial T^*} f(z)\,dz$$

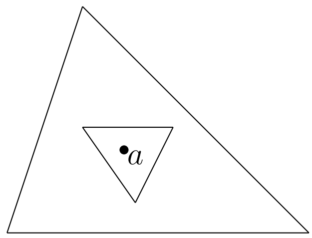


Figure 3: Point a contained in an inner triangle T^*.

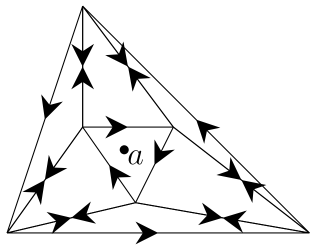


Figure 4: Integrating f around the subtriangles *except* T^*.

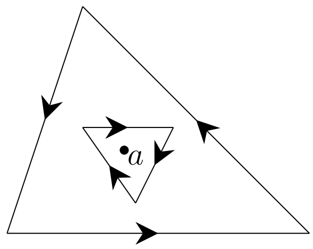


Figure 5: Internal edges cancel. We are left with ∂T and ∂T^* in opposite orientations.

So

$$\begin{aligned}\left|\int_{\partial T} f(z)\,dz\right| &= \left|\int_{\partial T^*} f(z)\,dz\right| \\ &\leq \text{length}(\partial T^*)\cdot \sup_{z\in\partial T^*} |f(z)|\end{aligned}$$

We have $T^* \subset B(a;\delta_\epsilon)$ so $z \in T^*$ implies $|f(z) - f(a)| < \epsilon$. Then $|f(z)| \leq |f(z) - f(a)| + |f(a)| < \epsilon + |f(a)|$. So $\sup_{z\in\partial T^*} \leq (\epsilon + |f(a)|)$. We can take ϵ arbitrarily small and δ arbitrarily small so length$(\partial T^*) \to 0$ and therefore $|\int_{\partial T} f(z)\,dz| = 0$.

In the case of multiple points of non-holomorphicity in the interior of T we simply isolate each one with an arbitrarily small ball and triangle. Then follow the same argument as above.

□

Definition 2.12 (Convex Domain)
A convex domain is a domain U such that for any two points $z_1, z_2 \in U$ then the straight line $\{(1-t)\cdot z_1 + t\cdot z_2 : 0 \leq t \leq 1\}$ is a subset of U.

Open balls are clearly convex: $z_1, z_2 \in B(z_0; r)$ implies $|z_1 - z_0| < r$ and $|z_2 - z_0| < r$.

Now

$$\begin{aligned}|z_0 - [(1-t)\cdot z_1 + t\cdot z_2]| &\le (1-t)\cdot|z_0 - z_1| \quad &+ t\cdot|z_0 - z_2| \\ &< (1-t)\cdot r &+ t\cdot r \\ &= r\end{aligned}$$

Definition 2.13 (Star-shaped Domain)
A star-shapaed domain is a domain U with some base point $a_0 \in U$ such that for all $w \in U$ the line segment $\{(1-t)\cdot a_0 + t\cdot w : 0 \le t \le 1\}$ is a subset of U.

We note that every convex domain is a star-shaped domain: for a convex domain U we may take the base point to be any point $a_0 \in U$ then by definition the line segment $\{(1-t)\cdot a_0 + t\cdot w : 0 \le t \le 1\}$ will be a subset of U for all $w \in U$.

Lemma 2.14
Let U be a star-domain. If $f : U \to \mathbb{C}$ is continuous on all of U and holomorphic everywhere except finitely many points in U then there exists some holomorphic primitve $F : U \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in U$.

Proof. Since U is a star-domain, there exists a base point $a_0 \in U$ so that $z \in U$ implies the line segment $\gamma_z(t) := (1-t)\cdot a_0 + t\cdot z, 0 \le t \le 1$ is in U. So define

$$F(z) := \int_{\gamma_z} f(w)\,dw$$

Also, U being open implies there is some $\epsilon > 0$ such that $B(z;\epsilon) \subset U$. Consider some $\eta \in \mathbb{C}$ such that $z+\eta \in B(z;\epsilon)$. Let $\delta_\eta := (t-1)\cdot z + t\cdot(z+\eta), 0 \le t \le 1$. Since open balls are convex then $\delta_\eta \subset B(z;\epsilon) \subset U$. See Figure 6 for a sketch. Theorem 2.11 (Cauchy's Integral Theorem for Triangles) tells us

$$\int_{\gamma_z} f(w)\,dw + \int_{\delta_\eta} f(w)\,dw - \int_{\gamma_{z+\eta}} f(w)\,dw = 0$$

I.e.

$$F(z) + \int_{\delta_\eta} f(w)\,dw - F(z+\eta) = 0$$

So

$$\begin{aligned}\left|\frac{F(z+\eta)-F(z)}{\eta} - f(z)\right| &= \frac{1}{|\eta|}\left|\int_{\delta_\eta} f(w) - f(z)\,dw\right| \\ &\le \sup_{w\in\delta_\eta} |f(w) - f(z)|\end{aligned}$$

Since f is continuous then $\sup_{w\in\delta_\eta} |f(w)-f(z)| \to 0$ (and by construction $\eta \to 0$) as $\epsilon \to 0$. So $F'(z) = f(z)$.

$\square$

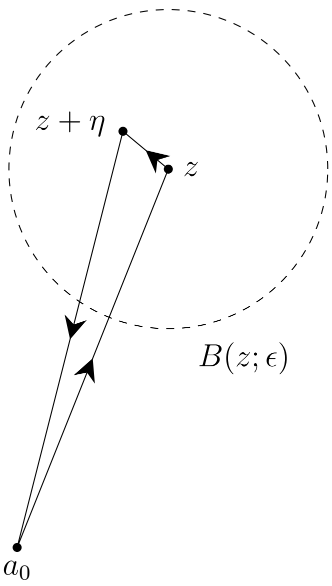


Figure 6: Definition of the primitive F using integrals along line segments in U.

Theorem 2.15 (Cauchy's Integral Theorem for Convex Domains)
Let U be a convex domain. If $f : U \to \mathbb{C}$ is continuous on all of U and holomorphic everywhere except finitely many points in U then for any closed path $\gamma \subset U$ we have

$$\int_\gamma f(z)\, dz = 0 \tag{7}$$

Proof. All convex domains are star-domains. So by Lemma 2.14 there exists a holomorphic primitive F such that $F'(z) = f(z)$ for all $z \in U$. Closed integrals of functions with primatives are 0. $\square$

Definition 2.16 (Homotopy)
Let γ, η be two paths in a domain U. We say that γ is homotopic to η if there exists a continuous function $\Gamma\colon [0,1] \times [0,1] \to U$ such that $\Gamma(t,0) = \gamma(t)$ and $\Gamma(t,1) = \eta(t)$ for all $t \in [0,1]$. For our purposes, we also require that for all $s \in [0,1]$ the map $t \mapsto \Gamma(t,s)$ is C^1 smooth.

It can be shown that homotopy is an equivalence relation.

Definition 2.17 (Simply connected)
A domain U is simply connected iff all pairs of paths γ and η with $\gamma(0) = \eta(0)$ and $\gamma(1) = \eta(1)$ are homotopic with the homotopy map Γ keeping the end points fixed: for all $s \in [0,1]$ we have $\Gamma(0,s) = \gamma(0) = \eta(0)$ and $\Gamma(1,s) = \gamma(1) = \eta(1)$.

Theorem 2.18 (Cauchy's Theorem for Simply Connected Domains)
If $U \subseteq \mathbb{C}$ is a simply connected domain then for any closed piecewise C^1 path γ in U

$$\int_\gamma f(z)\,dz = 0 \tag{8}$$

Proof. Split the path γ into two sections: let $\phi(t) := \gamma(\frac{t}{2}), 0 \leq t \leq 1$ and $\psi(t) := \gamma(1 - \frac{t}{2}), 0 \leq t \leq 1$. Note that ψ runs in the opposite direction to the corresponding section of γ so

$$\int_\gamma f(z)\,dz = \int_\phi f(z)\,dz - \int_\psi f(z)\,dz \tag{9}$$

with $\phi(0) = \gamma(0) = \gamma(1) = \psi(0)$ and $\phi(1) = \gamma(\frac{1}{2}) = \psi(1)$. Since U is simply connected then there exists a homotopy map $\Gamma : [0,1] \times [0,1] \to U$ such that $\Gamma(t,0) = \phi(t)$ and $\Gamma(t,1) = \psi(t)$.

It will be useful to prove that Γ is necessarily uniformly continuous. Let d denote the usual Euclidean metric on $[0,1] \times [0,1]$ and let $\epsilon > 0$. Since Γ is pointwise continuous by definition then for each $(x,y) \in [0,1] \times [0,1]$ there exists some $\delta_{(x,y)} > 0$ such that $(p,q) \in [0,1] \times [0,1]$ and $d((x,y),(p,q)) < \delta_{(x,y)}$ implies $|\Gamma(x,y) - \Gamma(p,q)| < \frac{\epsilon}{2}$. Now $\bigcup_{(x,y)\in[0,1]\times[0,1]} B((x,y); \frac{\delta_{(x,y)}}{2})$ is an open cover of $[0,1]\times[0,1]$ so by compactness we can find a finite subcover $[0,1]\times[0,1] = \bigcup_{j=1}^{J} B((x_j,y_j); \frac{\delta_j}{2})$ for some $J \in \mathbb{N}$. Let $\delta := \min(\{\delta_j\}_{j=1}^{J})$. Now for any $(x,y),(p,q) \in [0,1] \times [0,1]$ we have $(x,y) \in B((x_j,y_j); \frac{\delta_j}{2})$ for some $j \leq J$ so taking $d((x,y),(p,q)) < \frac{\delta}{2}$ implies

$$\begin{aligned} d((p,q),(x_j,y_j)) &\leq d((p,q),(x,y)) + d((x,y),(x_j,y_j)) \\ &< \frac{\delta}{2} + \frac{\delta_j}{2} \\ &\leq \frac{\delta_j}{2} + \frac{\delta_j}{2} \end{aligned}$$

so $|\Gamma(p,q) - \Gamma(x_j,y_j)| < \frac{\epsilon}{2}$. Now

$$\begin{aligned} |\Gamma(x,y) - \Gamma(p,q)| &\leq |\Gamma(x,y) - \Gamma(x_j,y_j)| + |\Gamma(x_j,y_j) - \Gamma(p,q)| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \end{aligned} \tag{10}$$

Since ϵ was arbitrary and we can use the same $\frac{\delta}{2}$ for all pairs of points $(x,y),(p,q)$ in $[0,1]\times[0,1]$ then Γ is uniformly continuous. Now we want to cover $\Gamma([0,1]\times[0,1])$ with a collection of open balls which are actually ball shaped — i.e. fully contained in U rather cut off by intersection with U. We will then subdivide our integral into a grid inside $\Gamma([0,1] \times [0,1])$ such that each grid cell is a subset of some ball — see figure 7.

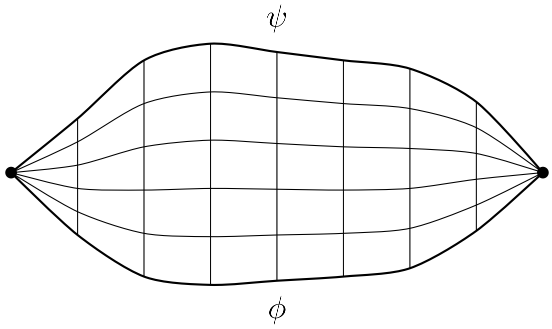


Figure 7: Image of the homotopy $\Gamma([0,1]\times[0,1])$ divided into a grid

Since U is open then for each $(x,y)\in[0,1]\times[0,1]$ there is some $R_{(x,y)}>0$ such that $D(\Gamma(x,y);R_{x,y}) := \{z\in\mathbb{C} : |z-\Gamma(x,y)|<r\}\subset U$. We have $\bigcup_{(x,y)\in[0,1]\times[0,1]} D(\Gamma(x,y);\frac{R_{(x,y)}}{2})$ is an open cover of $\Gamma([0,1]\times[0,1])$ so there exists a finite subcover $\bigcup_{l=1}^{L} D(\Gamma(x_l,y_l);\frac{R_l}{2})$ for some finite $L\in\mathbb{N}$. Let $R := \min(\{R_l\}_{l=1}^{L})$. Now for each $(x,y)\in[0,1]\times[0,1]$ we have $|\Gamma(x,y)-\Gamma(x_l,y_l)|<\frac{R}{2}$ for some $l<L$. Taking $z\in\mathbb{C}$ with $|z-\Gamma(x,y)|<\frac{R}{2}$ then

$$\begin{aligned} |z-\Gamma(x_l,y_l)| &\le |z-\Gamma(x,y)|+|\Gamma(x,y)-\Gamma(x_l,y_l)| \\ &< \frac{R}{2}+\frac{R}{2} \end{aligned} \tag{11}$$

So $D(\Gamma(x,y);\frac{R}{2})\subset D(\Gamma(x_j,y_j);R)\subset U$. Since our choice of $(x,y)\in[0,1]\times[0,1]$ was arbitrary then for all $(x,y)\in[0,1]\times[0,1]$ we have the $D((x,y);\frac{R}{2})$ ball fully contained in U.

Now we want to choose $\{(t_n,s_m)\}_{n,m=0}^{N,M}$ for some finite $N,M\in\mathbb{N}$ such that

$$\Gamma(t_{n+1},s_{m+1}),\Gamma(t_{n+1},s_m),\Gamma(t_n,s_{m+1})\in D(\Gamma(t_n,s_n);\frac{R}{2})\subset U \tag{12}$$

for all $n\le N-1$ and $m\le M-1$. This will define our grid points inside $\Gamma([0,1]\times[0,1])$. Since we have already established Γ is uniformly continuous then there exists $r>0$ such that for all $(x,y),(p,q)\in[0,1]\times[0,1]$ then $d((x,y),(p,q))<r$ implies $|\Gamma(x,y)-\Gamma(p,q)|<\frac{R}{2}$ where R is defined above. Pick $N=M>\frac{2}{r}$ and let

$$t_n := \frac{n}{N}, \quad s_m := \frac{m}{M}$$

This choice satisfies equation (12).

Notice that since $t_0=0$ we have $\Gamma(t_0,s_{m+1})=\Gamma(t_0,s_m)=\gamma(0)$ and $t_N=1$ so $\Gamma(t_{n+1},s_{m+1})=\Gamma(t_{n+1},s_m)=\gamma(1)$ for all $m\le M$. These grid cells actually only contain three vertices as in figure 7.

By equicontinuity of Γ we have that each cell is contained in an open ball of radius $\frac{R}{2}$. We may apply Theorem 2.15 (Cauchy's Integral Theorem for Convex Domains) so each integral of $f(z)$ around a grid cell equals 0. Therefore their sum is zero. The internal paths of the grid are each traversed twice in opposite directions so they cancel in the sum and we are left with ϕ and ψ traversed in opposite directions. So

$$\int_\phi f(z)\,dz - \int_\psi f(z)\,dz = 0$$

Then by equation (9)

$$\int_\gamma f(z)\, dz = 0$$

$\square$

Theorem 2.19 (Cauchy's Integral Formula)
Let U be a domain and $f : U \to \mathbb{C}$ be holomorphic. Suppose the closed ball $\overline{B(z_0; r)} \subset U$ for some $z_0 \in U$ and $r > 0$. Define the path $\gamma := z_0 + re^{it}, 0 \leq t \leq 2\pi$. Then for all z in the open ball $B(z_0; r)$ we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw \tag{13}$$

Proof. Firstly note that we can find a slightly larger open ball in U which contins the closed ball $\overline{B(z_0; r)}$. For each $z \in \overline{B(z_0; r)}$ there exists $\epsilon_z > 0$ such that $B(z; \epsilon_z) \subset U$. We can take $\bigcup_{z \in \overline{B(z_0;r)}} B(z; \epsilon_z)$ as an open cover of $\overline{B(z_0; r)}$ which is compact so there exist an $N \in \mathbb{N}$ and finite subcover $\bigcup_{n=1}^{N} B(z_n; \epsilon_n)$. Let $\epsilon := \min(\{\epsilon_n\}_{n=1}^{N}$. Now $|z - z_0| \leq r$ implies that for some $n \leq N$

$$\begin{aligned} |z - z_0| &\leq |z - z_n| + |z_n - z_0| \\ &< \epsilon + r \end{aligned}$$

so $\overline{B(z_0; r)} \subset B(z_0; r + \epsilon) \subset U$. Now Define

$$g(w) := \begin{cases} \frac{f(w) - f(z)}{w - z} & w \neq z \\ f'(z) & w = z \end{cases}$$

Note that g is continuous everywhere and holomorphic everywhere except possibly $w = z$ but this is inside disc. So we may apply Theorem 2.15 (Cauchy's Integral Theorem for Convex Domains) using the domain $B(z_0; r + \epsilon)$ to g:

$$\int_\gamma \frac{f(w)}{w - z}\, dw - \int_\gamma \frac{f(z)}{w - z}\, dw \equiv \int_\gamma g(w)\, dw = 0 \tag{14}$$

Notice that

$$\begin{aligned} \frac{1}{w - z} &= \frac{1}{w - z_0} \cdot \frac{1}{1 - \left(\frac{z - z_0}{w - z_0}\right)} \\ &= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}} \end{aligned}$$

which converges uniformly on $\partial\overline{B(z_0; r)} := \{z \in \mathbb{C} : |z - z_0| = r\}$ by the Weierstrass M-test (Conway, 1995, p.29) since $|z - z_0| < |w - z_0|$. We have $f(z)$ fixed and uniform convergence allows us to move the limit of the series outside the integral so equation (14) becomes

$$\int_\gamma \frac{f(w)}{w - z}\, dw = f(z) \sum_{n=0}^{\infty} \int_\gamma \frac{(z - z_0)^n}{(w - z_0)^{n+1}}\, dw$$

By Proposition 2.7 only the $n = 0$ term is non-zero, taking the value $2\pi i$. $\square$

2.3 Using Cauchy's Integral Formula

Cauchy's Integral Formula is extremely useful and is the basis for many proofs.

Corollary 2.20
Let $f : B(z_0, R) \to \mathbb{C}$ be holomorphic. Then f has a convergent power series representation

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad = \frac{f^{(n)}(z_0)}{n!}$$

for any $0 < r < R$. Notice that this tells us that holomorphic functions are infinitely differentiable — denoted C^∞.

For proof (relying on Cauchy's Integral Formula) see (Conway, 1995, p.86).

Definition 2.21 (Analytic)
A complex function f is analytic at a point z_0 if there is some $\epsilon > 0$ and complex constants $\{c_n\}_{n=0}^{\infty}$ such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)$$

and the series converges for $z \in B(z_0; \epsilon)$.

A complex function f is analytic on a domain U if it is analytic at every point $z_0 \in U$.

Corollary 2.20 tells us that all holomorphic functions are analytic. It can also be shown (Conway, 1995, pg. 35) that analytic functions are holomorphic. In practice, much of the literature uses the terms analytic and holomorphic interchangeably and many sources use opposing definitions to those used here.

Lemma 2.22
Let $U \subseteq \mathbb{C}$ be a domain and $f\colon U \to \mathbb{C}$ be continuous. Moreover, suppose

$$\int_\gamma f(z)\, dz = 0 \tag{15}$$

for any closed piecewise C^1-smooth path γ in U. Then f has a holomorphic primitive F such that $F' = f$.

Proof. Choose some $a_0 \in U$. For any $w \in U$ there exists a continuous map $\tilde{\gamma}_w\colon [0,1] \to U$ such that $\tilde{\gamma}_w(0) = a_0$ and $\tilde{\gamma}_w(1) = w$. Since U is open, for all x in the image of $\tilde{\gamma}_w$ there is some $\epsilon(x) > 0$ such that $B(x, \epsilon(x)) \subseteq U$.

The image of the compact set $[0,1]$ under the continuous map $\tilde{\gamma}_w$ is necessarily compact so there exists some finite collection $\{x_i\}$ such that $\{B(x_i, \epsilon(x_i))\}$ is a cover. Now we can define the path γ_w as the collection of straight lines joining each x_i where each line is contained within the union of balls at its endpoints and is therefore contained in U. Since γ_w is piecewise smooth we may define

$$F(w) = \int_{\gamma_w} f(z)\, dz \tag{16}$$

Suppose there is some other piecewise C^1-smooth path η_w starting at a_0 and ending at w. Then $\gamma_w - \eta_w$ forms a closed piecewise C^1-smooth path in U so by hypothesis

$$\int_{\gamma_w - \eta_w} f(z)\, dz = 0 = \int_{\gamma_w} f(z)\, dz - \int_{\eta_w} f(z)\, dz \tag{17}$$

i.e. $F(w)$ truly is just a function of w and does not depend on which C^1-smooth path we construct it with.

Now consider $w + h \in B(w, \epsilon)$ for some ϵ and $F(w+h) = \int_{\gamma_w + \delta_h} f(z)\, dz = F(w) + \int_{\delta_h} f(z)\, dz$ where δ_h is the straight line between w and $w + h$. Note that δ_h is in $B(w, \epsilon)$. So

$$F(w+h) = F(x) + hf(w) + \int_{\delta_h} (f(z) - f(w))\, dz \tag{18}$$

Then

$$\begin{aligned} \left| \frac{F(w+h) - F(w)}{h} - f(w) \right| &= \frac{1}{|h|} \left| \int_{\delta_h} f(z) - f(w)\, dz \right| \\ &\leq \frac{1}{|h|} \operatorname{length}(\delta_h) \sup_{\delta_h} |f(z) - f(w)| \\ &= \sup_{\delta_h} |f(z) - f(w)| \end{aligned} \tag{19}$$

Since f is continuous, as $h \to 0$ then $f(z) - f(w) \to 0$. So F is holomorphic with derivative f. ☐

Theorem 2.23 (Morera's Theorem)
Let $U \subseteq \mathbb{C}$ be a domain. let $f \colon U \to \mathbb{C}$ be continuous such that

$$\int_{\gamma} f(z)\, dz = 0$$

For all piecewise-C^1 closed curves γ in U. Then f is holomorphic on U.

Proof. By lemma 2.22, we know that f has an anti-derivative F which is itself holomorphic. Corollary 2.20 tells us that holomorphic functions are infinitely differentiable so $f' = F''$. ☐

2.4 Existence of Complex Logarithms

Definition 2.24 (Exponential Map)
For $z \in \mathbb{C}$, we define

$$e^z \equiv \exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Theorem 2.25 (A Logarithm Exists on Simply Connected Domains)
If f is holomorphic and nowhere zero in a simply connected domain U then there exists an injective holomorphic function $\log$ *such that* $\exp(\log(f(z))) = f(z)$.

Proof. Define

$$g(z) := \frac{f'(z)}{f(z)}$$

which is holomorphic on U. For any closed path in $\gamma \subset U$, by Theorem 2.18 (Cauchy's Theorem for Simply Connected Domains) we have

$$\int_\gamma g(z)\, dz = 0$$

Now by Lemma 2.22 we have the existence of a holomorphic primitive G such that

$$G'(z) = g(z) = \frac{f'(z)}{f(z)}$$

So $f(z) \cdot e^{-G(z)}$ has zero derivative on all of U and is therefore constant on U. Choose a point $z_0 \in U$. By Remark 2.6, there are many possible choices of $z^* \in \mathbb{C}$ such that $e^{z^*} = f(z_0)$ so choose one. Now define

$$\log(f(z)) := G(z) - G(z_0) + z^* \tag{20}$$

and observe that

$$\begin{aligned} e^{\log(f(z))} &= e^{G(z) - G(z_0) + z^*} \\ &= e^{G(z)} \cdot f(z_0) \cdot e^{-G(z_0)} \\ &= e^{G(z)} \cdot f(z) \cdot e^{-G(z)} \\ &= f(z) \end{aligned}$$

Note that our function $\log(z)$ is defined up to a constant, which depends on our choice of z^* (but not z_0) which can differ by integer multiples of 2π — different choices of z^* are called different "branches" of the logarithm.

Each branch of the logarithm is injective as

$$\begin{aligned} \log(z_1) &= \log(z_2) \\ \implies \exp(\log(z_1)) &= \exp(\log(z_2)) \\ \implies z_1 &= z_2 \end{aligned}$$

□

Corollary 2.26 (A Square Root Exists on Simply Connected Domains)
If f is holomorphic and nowhere zero on a simply connected domain U then there exists an injective holomorphic function g such that

$$g(f(z))^2 = f(z) \tag{21}$$

Moreover we have $g(U) \cap -g(U) = \emptyset$

We will abuse notation and write $g(f(z)) = \sqrt{f(z)}$ but it is to be understood this is not unique.

Proof. By Theorem 2.25 we then we can take a branch of the logarithm. Now define

$$g(f(z)) \coloneqq e^{\frac{1}{2}\cdot\log(f(z))}$$

and we are done.

The function is injective as

$$\begin{aligned} g(z_1) &= g(z_1) \\ \implies g^2(z_1) &= g^2(z_2) \\ \implies \exp(\log(z_1)) &= \exp(\log(z_2)) \\ \implies z_1 &= z_2 \end{aligned}$$

Similarly $-g(U) \cap g(U) = \emptyset$ as

$$\begin{aligned} g(z_1) &= -g(z_2) \\ \implies g^2(z_1) &= g^2(z_2) \\ \implies \exp(\log(z_1)) &= \exp(\log(z_2)) \\ \implies z_1 &= z_2 \\ \implies g(z_1) &= -g(z_1) && = 0 \\ \implies \exp(\log(z_1)) & && = 0 \\ \implies z_1 & && = 0 \quad \text{Contradiction} \end{aligned}$$

□

3 Montel's Theorem

3.1 Topology in the Complex Plane

The Riemann Mapping theorem is all about holomorphic functions: the goal being to show that we can always find a (unique) bijective holomorphic function from any open simply connected set $U \subset \mathbb{C}$ to the open disc.

In the proof we shall consider a collection (family) of candidate functions into the disc and go on to show that among these, we can always find at least one which is bijective (uniqueness is determined seperately). It is therefore helpful to consider the topology of complex function spaces. In particular we will develop a criterion — *normality* — to determine if a family functions is compact.

Compactness will allow us to reach a bijective function via a limiting process among the candidates. For the avoidance of ambiguity, we recall some basic definitions before proving some useful general topological statements.

Definition 3.1 (Compact)
A topological space $(X, \mathcal{U})$ is compact if every open cover has a finite subcover.

Definition 3.2 (Sequentially Compact)
A metric space (X, d) is sequentially compact *if every sequence in X has a convergent subsequence, with the limit necessarily being in X.*

Proposition 3.3
Compactness and sequential compactness are equivalent for metric spaces (with the induced topology).

It is not immediately clear how to proceed with the proof of Proposition 3.3 — compactness refers to properties of open covers and sequential compactness refers to properties of sequences. We need two more intermediate ideas as stepping stones.

Definition 3.4 (Limit Point)
Consider the metric space (X, d). If $A \subseteq X$ then a point $x \in X$ is a limit point of A if there is a sequence of distinct points $\{x_n\}$ in A such that $x = \lim x_n$.

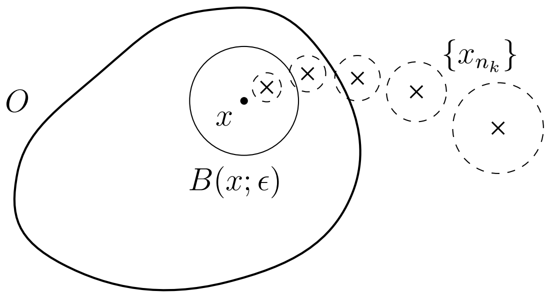
Definition 3.5 (Totally Bounded)
A metric space X is totally bounded if for every $\epsilon > 0$ there is a finite collection $\{x_n\} \subset X$ such that $X = \bigcup_{k=1}^{n} B(x_k; \epsilon)$.

We will also need the following lemma.

Lemma 3.6 (Lebesgue's Covering Lemma)
If (X,d) is a sequentially compact metric space and $\mathcal{O}$ is an open cover of X then there exists an $\epsilon > 0$ such that for all $x \in X$ there exists a set $O \in \mathcal{O}$ with $B(x;\epsilon) \subset O$.

Proof. Suppose, for contradiction, that $\mathcal{O}$ is an open cover of X and no such $\epsilon > 0$ can be found. In particular, for every integer n there is a point $x_n \in X$ such that $B(x_n; \frac{1}{n})$ is not contained in any set O in $\mathcal{O}$. Consider the sequence $\{x_n\}$ of these points. Since X is sequentially compact, there is a point $x \in X$ and a subsequence $\{x_{n_k}\}$ such that $x = \lim x_{n_k}$. Let $O \in \mathcal{O}$ such that $x \in O$ and choose $\epsilon > 0$ such that $B(x;\epsilon) \in O$. Now let N be such that $d(x, x_{n_k}) < \frac{\epsilon}{2}$ for all $n_k > N$. Let n_i be any integer larger than both N and $\frac{2}{\epsilon}$ and let $y \in B(x, \frac{1}{n_i})$. Then $d(x,y) \leq d(x, x_{n_i}) + d(x_{n_i}, y) < \frac{\epsilon}{2} + \frac{1}{n_i} < \epsilon$. I.e. $B(x_{n_i}; \frac{1}{n_i}) \subset B(x;\epsilon) \subset O$, contradicting the choice of x_n. □

The following diagram visualises the proof



Theorem 3.7
For a metric space X, the following are all equivalent:

(a) X is compact

(b) Every infinite set in X has a limit point

(c) X is sequentially compact

(d) X is complete and totally bounded

For brevity, we will only prove the following chain of implications. For the other implications, readers may consult e.g. Conway (Chapter 2). Combined with the converse (which is much more straightforward to prove) this proves Proposition 3.3.

$$\begin{tikzcd} & & & \& \arrow[dlll, bend right] & \\ a & b & c \arrow[ur] \arrow[rr] & & d \arrow[ul] \end{tikzcd}$$

Proof.
$(c) \implies (d)$

We will not directly use completeness but it follows straightforwardly. Suppose $\{x_n\}$ is a Cauchy sequence. By sequential compactness, there exists some subsequence $\{x_{n_k}\}$ which converges to a point x. By the Cauchy property, $\{x_n\}$ must also converge to x.

Suppose, for contradiction, that X is not totally bounded. Then there exists some $\epsilon > 0$ such that no finite collection $\{B(x_n; \epsilon)\}$ can cover X. We construct a sequence $\{x_n\}$ by first taking x_1 freely from X, then x_2 from $X - B(x_1; \epsilon)$, then x_3 from $X - (B(x_2; \epsilon) \cup B(x_1; \epsilon))$ etc. The sequence $\{x_n\}$ admits no convergent subsequence since all terms are greater than ϵ apart. This contradicts the hypothesis that X is sequentially compact.

$(c)\&(d) \implies (a)$

Let $\mathcal{O}$ be an open cover of X. By Lebesgue's Covering Lemma (lemma 3.6), there exists $\epsilon > 0$ such that for each $x \in X$, some $O \in \mathcal{O}$ contains $B(x; \epsilon)$. Since X is totally bounded, there exists a finite collection x_n such that $X = \bigcup_{n=1}^{N} B(x_n; \epsilon)$. We know that each $B(x_n; \epsilon)$ is contained in some $O_n \in \mathcal{O}$. Therefore $X = \bigcup_{n=1}^{N} O_n$ — a finite subcover.

□

3.2 Normal Families

Definition 3.8 (Normal Family v1)
A set (family) $\mathcal{F}$ of functions from some subset (open?) $U \subseteq \mathbb{C}$ to $\mathbb{C}$ is called a normal family if every sequence of functions $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges uniformly to some function f on every compact subset of U. The limiting function f is not necessarily in the family $\mathcal{F}$.

We would like to show that for a normal family $\mathcal{F}$, its closure $\mathcal{F}^{-}$ is necessarily compact. To do so, we will introduce a metric such that uniform convergence on every compact subset of the domain is equivalent to convergence with respect to our chosen metric. We will then appeal to proposition 3.3 to conclude that the closure $\mathcal{F}^{-}$ is compact with respect to the topology induced by the metric.

Of course, we ought not to have started the preceding paragraph discussing the closure $\mathcal{F}^{-}$ without first specifying a topology!

Before we can specify an appropriate metric (and discuss the equivalent metric free topology), we must construct a new way of covering the domain U — exhaustion by compact subsets.

Definition 3.9 (Interior of a set)
If a set A is a subset of some topological space $(X, \mathcal{O})$, then we define the interior of A to be the largest open subset of A. Concretely $\operatorname{int} A := \bigcup\{O \in \mathcal{O} | O \subset A\}$.

Definition 3.10 (Exhaustion (by compact subsets))
An exhaustion (by compact sets) of a set U is a sequence $\{K_n\}$ of compact subsets of U such that $U = \bigcup_{n=1}^{\infty} K_n$ and the following two conditions hold:

(a) $K_n \subset \operatorname{int} K_{n+1}$

(b) $K \subset U$ and K compact implies $K \subset K_n$ for some n

If such a collection exists, we say that U is exhaustible by compact sets.

Proposition 3.11
If U is an open subset of $\mathbb{C}$ then U is exhaustible by compact sets.

Proof. Let U be an open subset of $\mathbb{C}$. We shall explicitly construct an exhaustion: for each integer n take

$$K_n = \{z \in U : |z| \leq n\} \cap \left\{z \in U : d(z, \mathbb{C}\backslash U) \geq \frac{1}{n}\right\}$$

where $d(z_1, z_2)$ is the standard euclidean distance function. Since each K_n is bounded and closed (intersection of two closed sets) then it is compact.

Since U is open then for any $z \in U$ we can find an ϵ such that $B(z;\epsilon) \in U$ so $d(z, \mathbb{C}\backslash U) \geq \epsilon$. By the Archimedean property we can find $m > \max(|z|, \frac{1}{\epsilon})$ so $z \in K_m$ and we conclude $U = \bigcup_{n=1}^{\infty} K_n$.

Observe that the set

$$L_{n+1} = \{z \in U : |z| < n+1\} \cap \left\{z \in U : d(z, \mathbb{C}\backslash U) > \frac{1}{n+1}\right\}$$

is open and contained in K_{n+1}. Therefore $L_{n+1} \subset \text{int } K_{n+1}$. Since $K_n \subset L_{n+1}$ we conclude $K_n \subset \text{int } K_{n+1}$ i.e. property (a) is satisfied.

Since $U = \bigcup_{n=1}^{\infty} K_n$ property (a) tells us that likewise $U = \bigcup_{n=1}^{\infty} \text{int } K_n$. This forms an open cover for any compact subset $K \subset U$ so only finitely many int K_n are needed to cover K — take the largest n and conclude $K \subset K_n$ so property (b) is satisfied.

Since property (b) is implied by property (a), many authors (e.g. Lee: "Topological Manifolds") do not explicitly list property (b) in the definition of exhaustion by compact sets. □

3.3 A Metric for Uniform Convergence on Compact Sets

We are now in a position to construct an appropriate metric, with respect to which the closure $\mathcal{F}^-$ of a normal family $\mathcal{F}$ is necessarily compact. Let U be an open subset of $\mathbb{C}$ and let K_n be the exhaustion by compact sets as constructed above. For each K_n define the following metric on functions $f, g \in \mathcal{F}$:

$$\rho_n(f, g) = \sup\{|f(z) - g(z)| : z \in K_n\} \tag{22}$$

From these we construct the metric we have been seeking:

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{\rho_n(f, g)}{1 + \rho_n(f, g)} \cdot \left(\frac{1}{2}\right)^n \tag{23}$$

For proof that equations 22 and 23 do indeed define valid metrics (identity of indiscernibles, positivity, symmetry, triangle-inequality), the reader is directed to Chapter VII of Conway.

We will now show that uniform convergence of functions on compact subsets of $\mathbb{C}$ is exactly equivalent to convergence with respect to the metric given by equation 23.

Proposition 3.12
Take U to be an open subset of $\mathbb{C}$ and $\mathcal{F}$ a normal family on U. Given an $\epsilon > 0$ we may find a $\delta > 0$ and a compact subset $K \subset U$ such that for $f, g \in \mathcal{F}$ $\sup\{|f(z) - g(z)| : z \in K\} < \delta \implies \rho(f, g) < \epsilon$.

Conversely, given a $\delta > 0$ and a compact set K there is some $\epsilon > 0$ such that $\rho(f, g) < \epsilon \implies \sup\{|f(z) - g(z)| : z \in K\} < \delta$.

Proof.
Suppose ϵ is given. Consider the partial sum

$$\sum_{n=p+1}^{M} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}^{p+1} - \frac{1}{2}^M}{1 - \frac{1}{2}} \to \frac{1}{2}^p \quad \text{as } M \to \infty \tag{24}$$

Likewise

$$\sum_{n=1}^{p} \left(\frac{1}{2}\right)^n = 1 - \frac{1}{2}^p < 1 \tag{25}$$

Choose p such that $\frac{1}{2}^p < \frac{\epsilon}{2}$ and choose $\delta < \frac{\epsilon}{2}$. Choose $K = K_p$ from the exhaustion used in equation 23. Now $\rho_p(f, g) < \frac{\epsilon}{2}$ and since $K_n \subset K_{n+1}$ then $\rho_n(f, g) < \frac{\epsilon}{2}$ for $n \leq p$. Since $\frac{\rho_n(f,g)}{1+\rho_n(f,g)} < \rho_n(f, g)$ then by equation (25) we can see

$$\sum_{n=1}^{p} \frac{\rho_n(f, g)}{1 + \rho_n(f, g)} \cdot \left(\frac{1}{2}\right)^n < \frac{\epsilon}{2} \cdot \sum_{n=1}^{p} \left(\frac{1}{2}\right)^n < \frac{\epsilon}{2} \tag{26}$$

Also observe that $\frac{\rho_n(f,g)}{1+\rho_n(f,g)} < 1$ so equation (24) tells us

$$\sum_{n=p+1}^{\infty} \frac{\rho_n(f, g)}{1 + \rho_n(f, g)} \cdot \left(\frac{1}{2}\right)^n < \left(\frac{1}{2}\right)^p < \frac{\epsilon}{2} \tag{27}$$

Thus $\rho(f, g) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

For the converse, suppose $\delta > 0$ and a compact subset $K \subset U$ are given. Suppose, for the contrapositive, that $\sup\{|f(z) = g(z) : z \in K\} \geq \delta$. The exhaustion provides some compact set K_n such that $K \subset K_n$ so $\rho_n(f, g) \geq \delta$. Then $\rho(f, g) \geq \frac{\delta}{1+\delta} \cdot \left(\frac{1}{2}\right)^n$. Let $\epsilon = \frac{\delta}{1+\delta} \cdot \left(\frac{1}{2}\right)^n$. □

We now see that, for a set which admits an exhaustion, uniform convergence of sequences with respect to compact subsets is equivalent to convergence with respect to the metric defined in equation (23). This allows us to give an equivalent definition of a normal family of functions:

Definition 3.13 (Normal Family v2)
A set (family) $\mathcal{F}$ of functions from some open subset $U \subseteq \mathbb{C}$ to $\mathbb{C}$ is called a normal family if every sequence of functions $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges to some function f according to the metric specified by equation (23). The limiting function f is not necessarily in the family $\mathcal{F}$.

When defining the metric in equation (23), we explicitly used the exhaustion constructed in the proof of proposition 3.11. However, in the proof of proposition 3.12 we only used the properties specified in the definition of an exhaustion by compact subsets. For the metric to be well defined, we were forced to choose a specific exhaustion. The following corollary tells us that the topology does not depend on our choice of exhaustion.

Corollary 3.14
A subset of O a set of functions $\mathcal{F}$ from U to $\mathbb{C}$ is open with respect to the metric (23) if and only if for each $f \in O$ there is a compact set $K \subset U$ and a $\delta > 0$ such that $O \supset \{g : |f(z) - g(z)| < \delta, z \in K\}$.

Proof. If the set O is open with respect to the metric (23) then for every $f \in O$ there is some ϵ such that $B(f;\epsilon) \subset O$. The first part of proposition 3.12 tells us we may choose a $\delta > 0$ and subset $K \subset U$ so that $\{g : |f(z) - g(z)| < \delta, z \in K\} \subset B(f;\epsilon)$.

Conversely, if $\{g : |f(z) - g(z)| < \delta, z \in K\} \subset O$ for some compact set K, the second part of proposition 3.12 tells us there exists an ϵ so that $B(f;\epsilon) \subset \{g : |f(z) - g(z)| < \delta, z \in K\}$. □

The motivation for defining this topology (by first constructing the metric in equation (23)) was to prove that the closure of a normal family is necessarily compact. We need one final ingredient before we can prove this:

Lemma 3.15
The closure $\mathcal{F}^-$ of a normal family $\mathcal{F}$ is itself a normal family.

Proof. Suppose g_n is some sequence of functions in the closure $\mathcal{F}^-$ of a normal family $\mathcal{F}$. By well known properties of the closure (Munkres, 1999, p.97) we may approximate each function in the sequence g_n with a corresponding function $f_n \in \mathcal{F}$ that is within distance $\frac{1}{n}$ of g_n. Since f_n is a sequence from a normal family, it contains a subsequence f_{n_k} which converges to a limiting function f. Correspondingly g_{n_k} is a subsequence of g_n and must also converge to f. □

Theorem 3.16
The closure $\mathcal{F}^-$ of a normal family $\mathcal{F}$ is compact with respect to the metric specified in equation (23). Equivalently, $\mathcal{F}^-$ is compact with respect to the topology defined by corollary 3.14.

Proof. The closure $\mathcal{F}^-$ of a normal family $\mathcal{F}$ is itself normal by lemma 3.15. Since the closure contains all limit points, normality becomes equivalent to sequential compactness under the metric defined in equation (23). Proposition 3.3 tells us that in metric spaces, sequential compactness implies compactness. Thus $\mathcal{F}^-$ is compact. □

For all our hard work above to pay off, we need a convenient test to show that some family $\mathcal{F}$ of functions is in fact normal. For our purposes, the convenient test is uniform boundedness. We will show (Montel's Theorem) that if a family $\mathcal{F}$ of holomorphic functions on some open subset $U \subset \mathbb{C}$ is *uniformly bounded* on each compact subset of U then $\mathcal{F}$ is necessarily a normal family.

Definition 3.17 (Uniformly Bounded)
A family $\mathcal{F}$ of functions from some open set $U \subset \mathbb{C}$ to $\mathbb{C}$ is said to be uniformly bounded (on compact subsets) if for each compact subset $K \subset U$ there exists some $B > 0$ such that $|f(z)| \leq B$ for all $z \in K$ and all $f \in \mathcal{F}$.

3.4 Equicontintuity and The Arzela-Ascoli Theorem

Definition 3.18 (Equicontinuous)
A family $\mathcal{F}$ of functions from some open set $U \subset \mathbb{C}$ to $\mathbb{C}$ is equicontinuous at a point $z_0 \in U$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \epsilon$ for every $f \in \mathcal{F}$.

A family $\mathcal{F}$ of functions from some open set $U \subset \mathbb{C}$ to $\mathbb{C}$ is equicontinuous over a set $E \subset U$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $z_1, z_2 \in E$ and $|z_1 - z_2| < \epsilon$ implies $|f(z_1) - f(z_2)| < \epsilon$ for every $f \in \mathcal{F}$.

Equicontinuity is in some sense "uniform (across the family) uniform continuity."

Lemma 3.19
Let (X_n, d) be a metric space for each integer n and let $X = \Pi_{n=1}^{\infty} X_n$ be their cartesian product. That is, elements $\xi \in X$ take the form $\xi = \{x_1, x_2, x_3, \ldots\}$ where each x_i is taken from X_i. For $\xi = \{x_n\}$ and $\eta = \{y_n\}$ in X we define the metric

$$\sigma(\xi, \eta) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} \cdot \left(\frac{1}{2}\right)^n \tag{28}$$

We make the following claims:

(a) Equation (28) is in fact a metric.

(b) If $\xi^k = \{x_{x_n^k}$ is in X then $\xi^k \to \xi = \{x_n\}$ iff $x_n^k \to x_n$ for each n.

(c) If each (X_n, d_n) is compact then (X, σ) is compact.

Proof.
Point (a) is claimed without proof.

Suppose $\sigma(\xi^k, \xi) < \epsilon$. Each term in the sum must be less than or equal to the total, so for each n we have

$$\frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \cdot \left(\frac{1}{2}\right)^n \leq \epsilon \tag{29}$$

Rearranges to

$$d_n(x_n^k, x_n) \leq \frac{2^n \epsilon}{1 - 2^n \epsilon}$$

For any fixed n, we can take large k and make $\epsilon \to 0$ so the above bound tends to 0.

For the converse, we replicate the proof of proposition 3.12 and split the sum (28) into two parts $\sum_{n=1}^{p}$ and $\sum_{n=p+1}^{\infty}$. We note that the second part is bounded above by $\left(\frac{1}{2}\right)^p$ so choose p such that this part is less than $\frac{\epsilon}{2}$. By hypothesis, for each n there is some J_n

such that $d_n(x_n^k, x_n) < \delta$ for $k > J_n$. We consider the finite collection $J_1 \dots J_p$ and from these take the maximum. Following the proof of proposition 3.12 we choose δ such that the $\sum_{n=1}^{p}$ part is less than $\frac{\epsilon}{2}$. So $d(\xi^k, \xi) < \epsilon$ for $k \geq \max(J_1 \dots J_p)$ as above.

Finally, suppose each (X_n, d_n) is compact. We will appeal to proposition 3.3 and use sequential compactness to imply compactness. Take $\xi^k = \{x_1^k, x_2^k, \dots\}$ to be a sequence indexed by k in X. Now for each n we have x_n^k is a sequence indexed by k in X_n. We appeal to proposition 3.3 again to conclude that because X_n is compact, it is therefore sequentially compact so that x_n^k has some convergent subsequence $x_n^{k_{i,n}}$ where the map from the subsequence index i to $k_{i,n}$ is distinct for each n. Herein lies a problem because we do not yet know if the intersection $\bigcap_{n=1}^{\infty}\{k_{i,n}\}_{i=1}^{\infty}$ is non-empty. That is to say, it is not clear that we can find terms in the subsequences where $k_{j,n}$ take the same value k for all n (but different j) and allow us to form a complete ξ^{k_i} term.

We follow Conway (although essentially the same diagonalisation argument is used by Ahlfors and Stein/Shakarchi directly in the proof of Arzela-Ascoli/Montel's Theorem) and construct the x_n^k subsequences inductively. First consider $\{x_1^k\}_{k=1}^{\infty}$ as a sequence in X_1. Let $\{x_1^{k_1(i)}\}_{i=1}^{\infty}$ be the convergent subsequence in X_1. Now $\{x_2^{k_1(i)}\}_{i=1}^{\infty}$ is a sequence in X_2 so there exists a convergent subsequence $\{x_2^{k_1\circ k_2(i)}\}_{i=1}^{\infty}$. We continue inductively noting that $\{x_n^{k_1\circ\dots\circ k_m(i)}\}_{i=1}^{\infty}$ remains convergent for all $m \leq n$ (any subsequence of a convergent sequence is convergent) and that each $k_n : \mathbb{N} \to \mathbb{N}$ is monotonic by definition of subsequence. We now construct the subsequence $\{\xi^{k(j)}\}_{j=1}^{\infty}$ by setting $k(j) = k_1 \circ \dots \circ k_j(j)$ and claim this is convergent. Since we have already proved claim (b) we will use this and simply show that $\{\xi^{k(j)}\}_{j=1}^{\infty}$ converges pointwise.

Choose a fixed n. Since $\{x_n^{k_1\circ\dots\circ k_n(i)}\}_{i=1}^{\infty}$ is convergent by construction, we can find some I so that $i > I$ implies closeness to the limit x_n. Take $i > \max(I, n)$ and consider the n'th coordinate in $\xi^{k(i)}$ — namely $x_n^{k_1\circ\dots\circ k_i(i)}$. Since $i > n$ this equals $x_n^{k_1\circ\dots\circ k_n(l)}$ where $l = k_{n+1} \circ \dots \circ k_i(i)$. Monotonicity tells us $l \geq i$ so the n'th coordinate of $\xi^{k(i)}$ is close to the limit x_n for $i > I$. Now we have proved (X, σ) is sequentially compact, we appeal once more to proposition 3.3 and conclude that (X, σ) is compact. □

Lemma 3.20
Suppose $\mathcal{F}$ is a family of continuous functions from the open set $U \subset \mathbb{C}$ to $\mathbb{C}$ and it is equicontinuous at each point of U. Then $\mathcal{F}$ is equicontinuous over each compact subset $K \subset U$.

Proof. Take K a compact set. Since $\mathcal{F}$ is equicontinuous at each point in U then for every $z \in K$ there is some $\delta_z > 0$ such that $|z - w| < \delta_z$ implies $|f(z) - f(w)| < \frac{\epsilon}{2}$ for all $f \in \mathcal{F}$. The collection $B(z; \delta_z)$ forms an open cover of K. By proposition 3.3 K is sequentially compact so we may apply lemma 3.6 (Lebesgue's Covering Lemma): there exists a δ such that for all $w \in K$ then $B(w; \delta) \subset B(z; \delta_z)$ for some $B(z; \delta_z)$. Now for any $w_1, w_2 \in K$ with $|w_1 - w_2| < \delta$ then $w_2 \in B(w_1; \delta) \subset B(z; \delta_z)$ for some $z \in K$. Then $|z - w_1| < \delta_z$ and $|z - w_2| < \delta_z$ so $|f(w_1) - f(w_2)| \leq |f(z) - w_1| + |f(z) - w_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. □

Theorem 3.21 (Arzela-Ascoli)
A family of continuous functions from an open set $U \subset \mathbb{C}$ to $\mathbb{C}$ is normal if the following two conditions are satisfied:

(a) For each z in U the set $\{f(z) : f \in \mathcal{F}\}$ has compact closure in $\mathbb{C}$.

(b) The family $\mathcal{F}$ is equicontinuous at each point of U

Proof. The set D of all $z \in U$ with rational real and imaginary parts is dense in U. Morover, by diagonalization we can arrange this set into the sequence $\{z_n\}_{n=1}^{\infty}$. Let $X_n = \{f(z_n) : f \in \mathcal{F}\}^{-}$. Hypothesis (a) tells us this is a compact metric space (using the usual Euclidean metric in $\mathbb{C}$). Let $X = \prod_{n=1}^{\infty} X_n$. By lemma 3.19 then (X, σ) is a compact (and sequentially compact) metric space — where σ is the metric given by equation (28). Consider some sequence of functions $\{f_k\}_{k=1}^{\infty} \subset \mathcal{F}$. Define a corresponding sequence in X by setting $f_k^* = \{f_k(z_1), f_k(z_2), \ldots\}$. Since (X, σ) is sequentially compact then there is some subsequence $\{f_{k(i)}^*\}_{i=1}^{\infty}$ which converges to a limit $\xi = \{w_1, w_2, \ldots\}$. Define $g^* : D \to \mathbb{C}$ such that $g^*(z_n) = w_n$. Our goal is to extend g^* to some function g defined on U. We will then show that $f_{k(i)}$ converges uniformly to g on every compact subset $E \subset U$.

First we observe that for any compact subset $E \subset U$ the function $g*$ is uniformly continuous on $D \cap E$. Take $\epsilon > 0$. By hypothesis (b) and Lemma 3.20 we know that $\mathcal{F}$ is equicontinuous on E, so there exists $\delta > 0$ such that $u, v \in E$ and $|u - v| < \delta$ implies $|f(u) - f(v)| < \frac{\epsilon}{3}$ for all $f \in \mathcal{F}$. Lemma 3.19 tells us that for any $z \in D$ there exists J_z such that $i > J_z$ implies $|g^*(z) - f_{k(i)}(z)| < \frac{\epsilon}{3}$. Now take $u, v \in D \cap E$ with $|u - v| < \delta$ and let $i > \max(J_u, J_v)$ so

$$\begin{aligned} |g^*(u) - g^*(v)| &\leq |g^*(u) - f_{k(i)}(u)| + |f_{k(i)}(u) - f_{k(i)}(v)| + |f_{k(i)}(v) - g^*(v)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \end{aligned} \tag{30}$$

Since ϵ was arbitrary, we conclude g^* is uniform continuous on $D \cap E$.

Since D is dense then for any $z \in U$ we may choose some sequence $\{x_j\}_{j=1}^{\infty} \subset D$ such that $\{x_j\}_{j=1}^{\infty} \to z$. We define $g(z) = \lim\{g^*(x_j)\}_{j=1}^{\infty}$. To show that g exists, we must prove that $\{g^*(x_j)\}_{j=1}^{\infty}$ converges. Since this is a sequence in $\mathbb{C}$, which is complete, we will use Cauchy convergence.

We will only use g on compact subsets of U, so we assume $z, x_j \in E \subset U$ where E is compact. Strictly, we need to take x_j from a slightly larger compact set E' such that $E \subset E'$ as we want to approximate all points $z \in E$ with sequences in $E' \cap D$: to construct such an E' note that for all $z \in E$ there exists $r_z > 0$ such that $B(z; 2r_z) \subset U$. Now $\bigcup_{z \in E} B(z; r_z)$ is an open cover of E so there exists an $M \in \mathbb{N}$ and a finite subcover $E \subset \bigcup_{m=1}^{M} B(z_m; r_m)$. So let $E' := \bigcup_{m=1}^{M} \overline{B(z_m; r_m)}$ which is a finite union of compact sets so compact.

Take $\epsilon > 0$. Since g^* is uniformly continuous on $D \cap E'$ then there exists $\delta > 0$ such that $u, v \in D \cap E'$ and $|u - v| < \delta$ implies $|g^*(u) - g^*(v)| < \epsilon$. Since $\{x_j\}_{j=1}^{\infty} \to z$ we can find J such that $j, l > J$ implies $|x_j - z| < \frac{\delta}{2}$ and $|x_l - z| < \frac{\delta}{2}$ so $|x_j - x_l| < \delta$. Then $j, l > J$ implies $|g^*(x_j) - g^*(x_l)| < \epsilon$. Since ϵ was arbitrary we conclude $\{g(x_j)\}_{j=1}^{\infty}$ is Cauchy so the limit exists.

To prove uniqueness, we take a new sequence $\{y_j\}_{j=1}^{\infty} \to z$ and consider $\{g^*(y_j)\}_{j=1}^{\infty}$. By the above, it converges to some point $h(z)$. Again consider $\epsilon > 0$ and note that by uniform continuity of g^* then there exists $\delta > 0$ such that $u, v \in D \cap E'$ and $|u - v| < \delta$ implies $g^*(u) - g^*(v) < \frac{\epsilon}{3}$. Take J such that for $j > J$ we have $|y_j - z| < \frac{\delta}{2}$ and $|g^*(y_j) - h(z)| < \frac{\epsilon}{3}$ and $|x_j - z| < \frac{\delta}{2}$ and $|g^*(x_j) - g(z)| < \frac{\epsilon}{3}$. Then $|x_j - y_j| < \delta$. So

$$\begin{aligned} |h(z) - g(z)| &\leq |h(z) - g^*(y_j)| + |g^*(y_j) - g^*(x_j)| + |g^*(x_j) - g(z)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \end{aligned} \tag{31}$$

Since ϵ was arbitrary we conclude $h(z) = g(z)$ so g is uniquely defined on compact subsets of U. We note in passing that for all $z_n \in D$ then $g(z_n) = g^*(z_n)$ as we can consider the sequence $\{z_n, z_n, z_n, \ldots\} \to z_n$.

Morover g is uniformly continuous any compact subsets $E \subset U$. We have already shown that $\mathcal{F}$ is equicontinuous on E' and that g^* is equicontinuous on $D \cap E'$. Let $\epsilon > 0$. By equicontinuity of g^* then there exists δ such that for all $u, v \in D \cap E'$ then $|u - v| < \delta$ implies $|g^*(u) - g^*(v)| < \frac{\epsilon}{3}$. Consider $x, y \in E'$ with $|x - y| < \frac{\delta}{3}$. Since $D \cap E'$ is dense in E then there exist sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset D \cap E'$ such that $\{x_n\}_{n=1}^{\infty} \to x$ and $\{y_n\}_{n=1}^{\infty} \to y$. So for sufficiently large n we have $|x_n - x| < \frac{\delta}{3}$ and $|g^*(x_n) - g(x)| < \frac{\epsilon}{3}$ and $|y_n - y| < \frac{\delta}{3}$ and $|g^*(y_n) - g(y)| < \frac{\epsilon}{3}$. Then $|x_n - y_n| \leq |x_n - x| + |x - y| + |y - y_n| < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3}$. So

$$\begin{aligned} |g(x) - g(y)| &\leq |g(x) - g^*(x_n)| + |g^*(x_n) - g^*(y_n)| + |g^*(y_n) - g^*(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \end{aligned} \tag{32}$$

Again, ϵ was arbitrary so g is uniformly continuous on compact subsets of U.

Finally, we must show $f_k(i)$ converges uniformly to g on every compact subset of U. Let $E \subset U$ be compact. Take $\epsilon > 0$. There exists $\delta > 0$ such that $u, v \in E$ and $|u - v| < \delta$ implies $|g(u) - g(v)| < \frac{\epsilon}{3}$ and $|f(u) - f(v)| < \frac{\epsilon}{3}$ for all $f \in \mathcal{F}$. Recall that the dense set/sequence $D = \{z_n\}_{n=1}^{\infty}$ is the diagonalised collection of all complex numbers in U with real and imaginary parts. Then $\{B(z_n; \delta)\}_{n=1}^{\infty}$ is an open cover of E. By compactness, we can take a finite subcover with N elements. For convenience of notation we reindex the subcover as $\{z_n\}_{n=1}^{N}$. By Lemma 3.19, for each z_n in this finite collection, we can find J_n such that $i > J_n$ implies $|f_{k(i)}(z_n) - g^*(z_n)| < \frac{\epsilon}{3}$. Since N is finite, we can set $J = \max(J_1 \ldots J_N$ and take $i > J$. Now for any $z \in E$, then z is contained in some $B(z_n; \delta)$ so

$$\begin{aligned} |f_{k(i)}(z) - g(z)| &\leq |f_{k(i)}(z) - f_{k(i)}(z_n)| + |f_{k(i)}(z_n) - g(z_n)| + |g(z_n) - g(z)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \end{aligned} \tag{33}$$

Once more, ϵ (and E) was arbitrary so we conclude $\{f_{k(i)}\}_{i=1}^{\infty}$ converges uniformly to g on compact subsets of U. Since our original sequence $\{f_k\}_{k=1}^{\infty} \subset \mathcal{F}$ was also arbitrary, we conclude that $\mathcal{F}$ is normal by Definition 3.8. $\square$

3.5 Local Boundedness and Montel's Theorem

Definition 3.22 (Locally Bounded)
A set $\mathcal{F}$ of complex analytic functions on a domain U is locally bounded if for each point $z_0 \in U$ there is an $r > 0$ and $M < \infty$ such that

$$\sup\{|f(z)| : |z - z_0| < r, f \in \mathcal{F}\} = M \tag{34}$$

That is to say, around each point z_0 there is a disc on which $\mathcal{F}$ is uniformly bounded.

Theorem 3.23 (Montel's Theorem)
Suppose $\mathcal{F}$ is a family of holomorphic functions on a domain U. Then $\mathcal{F}$ is normal if it is locally bounded.

Note that the converse is also true but we shall not need it. For proof of the converse statement, see e.g. (Conway, 1995, p. 153).

Proof. We shall use Theorem 3.21 (Arzela-Ascoli).
Condition (a) is straightforwardly satisfied: for each $z \in U$ then local boundedness implies $\{f(z) : f \in \mathcal{F}\}$ is bounded so its closure is clearly compact.

To show that condition (b) of the Arzela-Ascoli Theorem is also satisfied, consider a point $z_0 \in U$. Local boundedness implies there is some $r > 0$ and $M < \infty$ such that $B(z_0; 2r) \subset U$ and $|f(z)| < M$ for all $z \in B(z_0, 2r)$ for all $f \in \mathcal{F}$. Then the closed set $\bar{B}(z_0; r) \subset B(z_0; 2r)$ satisfies the same properties. Let $|z - z_0| < \frac{1}{2}r$ and $f \in \mathcal{F}$. Using Cauchy's Integral Formula on the path $\gamma(t) = z_0 + re^{it}, 0 \leq t \leq 2\pi$ then

$$\begin{aligned}
|f(z_0) - f(z)| &= \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)}{w - z_0}\, dw - \int_\gamma \frac{f(w)}{w - z}\, dw\right| \\
&= \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(z_0 - z)}{(w - z_0)(w - z)}\, dw\right| \\
&\leq \frac{1}{2\pi}\int_\gamma \left|\frac{f(w)(z_0 - z)}{(w - z_0)(w - z)}\right| dw \\
&< \frac{2\pi r}{2\pi} \cdot \frac{M \cdot |z_0 - z|}{r \cdot \frac{r}{2}} = \frac{2M}{r} \cdot |z_0 - z|
\end{aligned} \tag{35}$$

Thus for any $z_0 \in U$ and $\epsilon > 0$ we can let $\delta < \min(\frac{r}{2}, \frac{r}{2M} \cdot \epsilon)$ and have $|z_0 - z| < \delta$ imply $|f(z_0) - f(z)| < \epsilon$ for all $f \in \mathcal{F}$. I.e. $\mathcal{F}$ is equicontinuous at every point in U.

Now $\mathcal{F}$ satisfies both conditions for the Arzela-Ascoli theorem so it must be a normal family. □

3.6 Some Convergence Results

Proposition 3.24
If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on some domain U and f is a continuous function on U such that $\{f_n\}_{n=1}^{\infty}$ converges to f with respect to the metric defined in equation (23) then f is also holomorphic. Moreover, necessarily $f_n^{(k)} \to f^{(k)}$ where (k) denotes the k'th derivative.

Proof. We first imitate the proof of Lemma 2.14 to establish the existence of a primitive for f. For any z_0 in U there exists $\epsilon > 0$ such that $B(z_0; \epsilon) \subset U$. This is a convex domain therefore also a star-domain. For any $z \in B(z_0; \epsilon)$ we define $\gamma_z := (1-t) \cdot z_0 + t \cdot z, 0 \leq t \leq 1$ and

$$F(z) := \int_{\gamma} f(w)\, dw$$

Recall that in the proof of Lemma 2.14, all that was needed to prove that F is holomorphic with $F'(z) = f(z)$ was that the closed integral around any triangle in the star-domain is necessarily zero. In this case, we know that any triangle $T \subset B(z_0; \epsilon)$ is closed and bounded therefore compact. So by Proposition 3.12 the functions $\{f_n\}_{n=1}^{\infty}$ converge uniformly on T to f. Uniform convergence allows us to "move" limits outside of integrals so

$$\int_{\partial T} f(w)\, dw = \int_{\partial T} \lim_{n\to\infty} f_n(w)\, dw = \lim_{n\to\infty} \int_{\partial T} f_n(w)\, dw = 0$$

where the last equality follows from Theorem 2.10 (Goursat's Theorem). Now by Corollary 2.20 $f' = F''$ on $B(z_0; \epsilon)$. Since z_0 was arbitrary then by definition f is holomorphic on U.

Now to prove convergence of the derivatives, consider again some arbitrary point $z_0 \in U$ with $2r > 0$ such that $B(z_0; 2r) \subseteq U$. Then the closed ball $\overline{B(z_0; r)} \subset U$. By a similar argument (take the union over of balls around all $z \in \overline{B(z_0; r)}$ and use the minimum radius of a finite subcover) to that used in the proof of Theorem 2.19 there exists $R > r$ such that $\overline{B(z_0; R)} \subset U$. Let $\gamma := z_0 + R \cdot e^{2\pi i \cdot t}, 0 \leq t \leq 1$ then by Corollary 2.20, for $z \in B(z_0; r)$ we have

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f_n(w) - f(w)}{(w-z)^{k+1}}\, dw$$

Let $M_n := \sup\{|f_n(w) - f(w)| : |w - z_0| = R\}$. Notice that $|w - z| > (R - r)$. So

$$\begin{aligned} |f_n^{(k)}(z) - f^{(k)}(z)| &\leq \frac{k!}{2\pi} \int_{\gamma} \left| \frac{f_n(w) - f(w)}{(w-z)^{k+1}} \right| dw \\ &< \frac{k! M_n R}{(R-r)^{k+1}} \end{aligned}$$

Since f_n converges uniformly to f on $\overline{B(z_0; R)}$ then for any $\delta > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $M_n < \delta$. So $f_n^{(k)}$ converges uniformly to $f^{(k)}$ on $B(z_0; r)$.

As $z_0 \in U$ was arbitrary then for every $z \in U$ there exists $r_z > 0$ such that $f_n^{(k)}$ converges uniformly to $f^{(k)}$ on $B(z; r_z)$. So for any compact set $E \subset U$ then cover it with $\bigcup_{z \in E} B(z; r_z)$. By compactness, there exists a finite subcover so for any $\delta > 0$ we can simply take the maximum $N \in \mathbb{N}$ such that $n > N$ implies $|f_n^{(k)}(z) - f^{(k)}(z)| < \delta$ for all z

in every ball constituting the subcover. I.e. f_n converges uniformly to f on all of U. By Proposition 3.12 this is equivalent to convergence with respect to the metric defined by equation (23).

$\square$

Proposition 3.25
The map $\Phi : f \to f'(z_0)$ sending a holomorphic function f to its derivative evaluated at some point z_0 in its domain is continuous with respect to the metric defined by equation (23).

Proof. We consider the set X of all holomorphic functions on a domain U. By Proposition 3.14 we will need to show that for every $f \in X$ for all $\epsilon > 0$ there exists a compact set $E \subset U$ and $\delta > 0$ such that for any $g \in X$ then $\sup\{|g(z) - f(z)| : z \in E\} < \delta$ implies $|g'(z_0) - f'(z_0)| < \epsilon$. This is the statement that the pre-image of every open set must be open for a mapping to be called continuous.

We follow the proof above for Proposition 3.24. By the same reasoning we are able to find some $r > 0$ such that the compact set $E := \overline{B(z_0; r)} \subset B(z_0; 2r) \subset U$. Now let $\gamma := z_0 + r \cdot e^{2\pi i \cdot t}, 0 \leq t \leq 1$ so by Corollary 2.20 we have

$$\begin{aligned} |g'(z_0) - f'(z_0)| &\leq \frac{1}{2\pi} \int_\gamma \left| \frac{g(w) - f(w)}{(w - z_0)^2} \right| dw \\ &\leq \frac{M}{r} \end{aligned} \tag{36}$$

Where $M := \sup\{|g(w) - f(w)| : w \in \overline{B(z_0; r)}\}$. So for any $\epsilon > 0$ we have $M < \frac{\epsilon}{r}$ implies $|g'(z_0) - f'(z_0)| < \epsilon$. $\square$

4 Hurwitz's Theorem

Hurwitz's Theorem will be used to show our desired mapping is in fact injective.

4.1 Rouché's Theorem

Theorem 4.1 (Rouché's Theorem)
Suppose f and g are holomorphic functions in a neighbourhood of the closed ball $\bar{B}(z_0; R)$ with no zeroes on the circular path $\gamma = \{z : |z - z_0| = R\}$. Let Z_f, Z_g be the number of zeroes of f and g inside the region bounded by γ. If

$$|f(z) + g(z)| < |f(z)||g(z)|$$

on γ, then

$$Z_f = Z_g$$

Strictly speaking we need to account for the "multiplicities" of the zeros. Since all holomorphic functions are analytic (Corollary 2.20), we are able to express any holomorphic function f in the form $f(z) = (z - z_1)^m \cdot h(z)$ for some $m \in \mathbb{N}$ and holomorphic h such that $h(z_1) \neq 0$. The multiplicity of the the zero of f at z_1 is then m and if z_1 is in $B(z_0; R)$ then this would add m to Z_f.

The proof relies on the Residue Theorem, which readily follows from the Cauchy Integral Formula and Cauchy's Theorem which we have already proved. For brevity we cite (Conway, 1995, 125).

4.2 Hurwitz's Theorem

Theorem 4.2 (Hurwitz' Theorem)
Let U be a domain and suppose the sequence of holomorphic functions $\{f_n\}_{n=1}^{\infty}$ converges to f with respect to the metric defined in equation (23). Equivalently, by Proposition 3.12, suppose $\{f_n\}_{n=1}^{\infty}$, converges uniformly on every compact subset of U. If the closed ball $\bar{B}(z_0; R)$ is a subset of U and $f(z) \neq 0$ for any z in $\gamma := \{z \in U : |z - z_0| = R\}$ then there exists N such that $n \geq N$ implies f and f_n have the same number of zeros in the open ball $B(z_0; R)$.

Proof. Since $f(z) \neq 0$ for any $z \in \gamma$ then

$$\delta := \inf\{|f(z)| : z \in \gamma\} > 0$$

Since γ is compact then $\{f_n\}_{n=1}^{\infty} \to f$ uniformly on γ. So there exists N such that $n \geq N$ implies $f_n(z) \neq 0$ for any $z \in \gamma$ and

$$|f(z) - f_n(z)| \leq \frac{\delta}{2} < |f(z)| < |f(z)| + |f_n(z)|$$

for $z \in \gamma$. Let $g_n := -f_n$ and apply theorem 4.1 (Rouché's Theorem) to conclude that f and g_n — and subsequently f_n — share the same number of zeros on the open ball $B(z_0; R)$. □

Corollary 4.3

Let U be a domain and suppose the sequence of holomorphic functions $\{f_n\}_{n=1}^{\infty}$ converges to f with respect to the metric defined in equation (23). Additionally, suppose $f_n(z) \neq 0$ for all $z \in U$ and $n \in \mathbb{N}$. Then f is either identically zero or $f(z) \neq 0$ for all $z \in U$.

Proof. By proposition 3.24, we know that f is holomorphic. Therefore it is either identically zero on U or the zeroes are isolated (this is readily proved using Corollary 2.20).

Suppose that f is not identically zero. Consider some point $z_0 \in U$. Since the zeros are isolated then for some $r > 0$ we have $f(z) \neq 0$ for all $z \in \{z \in U : 0 < |z - z_0| < 2R\} \subset U$. Moreover $f(z_0) \neq 0$ for all $z \in \gamma := \{z \in U : |z - z_0| = R\} \subset U$. Apply Theorem 4.2 (Hurwitz's Theorem) and conclude that $f(z_0) \neq 0$ since none of the f_n are zero anywhere inside the ball $B(z_0, R)$. Since z_0 was an arbitrary point in U, f is nowhere zero. $\square$

5 The Riemann Mapping Theorem

Theorem 5.1 (Riemann Mapping Theorem)
Given a simply connected proper subdomain $U \subsetneq \mathbb{C}$ and a base point $z_0 \in U$, there exists a unique bijective holomorphic function $f\colon U \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) \in \mathbb{R}$ with $f'(z_0) > 0$.

Proof.
To prove uniqueness, suppose that we have two conformal isomorphisms $\phi : U \to \mathbb{D}$ and $\psi : U \to \mathbb{D}$ with $\phi(z_0) = \psi(z_0) = 0$ and $\phi'(z_0) > 0$ and $\psi'(z_0) > 0$. Then $\phi \circ \psi^{-1} : \mathbb{D} \to \mathbb{D}$ is a conformal isomorphism from the disc to itself. The only conformal isomorphisms from the disc to itself are of the form (where $a \in \mathbb{D}$ and $\theta \in R$) (Conway, 1995, p.132):

$$M(z) := e^{i\theta}\frac{z-a}{1-\bar{a}z} \tag{37}$$

So $\phi\circ\psi^{-1} \equiv M$ for some a and $e^{i\theta}$. Since $\phi\circ\psi^{-1}$ maps 0 to 0 then $a = 0$ and $M(z) = e^{i\theta}z$. Now

$$\begin{aligned} e^{i\theta} = M'(0) &= \frac{1}{\psi'(\psi^{-1}(0))}\cdot\phi'(\psi^{-1}(0)) \\ &= \frac{1}{\psi'(z_0)}\cdot\phi'(z_0) > 0 \end{aligned} \tag{38}$$

So $e^{i\theta} = 1$ and M is the identity map. Then $\phi \equiv \psi$.

For proof of existence, consider the family of functions

$$\mathcal{F} := \{f \in \text{holomorphic functions on U} \,|\, f \text{ is injective, } f(z_0) = 0, f'(z_0) > 0, f(U) \subseteq \mathbb{D}\} \tag{39}$$

We make the following claims:

Claim 1: $\mathcal{F}$ is non-empty
Claim 2: $\mathcal{F}^{-} = \mathcal{F} \cup \{0\}$ (the closure of $\mathcal{F}$ is itself union the zero function)
Claim 3: There exists $f \in \mathcal{F}^{-}$ such that $f'(z_0) \geq g'(z_0)$ for all $g \in \mathcal{F}$
Claim 4: If $g \in \mathcal{F}$ and $g(U) \neq \mathbb{D}$ then there exists $h \in \mathcal{F}$ with $h'(z_0) > g'(z_0)$

Upon proving these claims, we can conclude that the family $\mathcal{F}$ has some function f with maximal derivative at the base point (claim 3) and that f is either a member of $\mathcal{F}$ or the zero function (claim 2). Since $\mathcal{F}$ is non-empty (claim 1) then f cannot be the zero function. Finally, we see that $f(U) = \mathbb{D}$ because otherwise it would not have maximal derivative at the base point (claim 4).

<u>Proof of claim 1:</u>
Since $U \subsetneq \mathbb{C}$ is simply connected, by Corollary 2.26 we may define an injective holomorphic square root function $h(z) := \sqrt{z-a}$.

Note that $h(U)$ is open so there exists $r > 0$ such that $\{w \in \mathbb{C} : |w - h(z_0)| < r\} \subset U$. Also, by Corollary 2.26, $w \in h(U)$ implies $-w \notin h(U)$. So $\{w \in \mathbb{C} : |-w - h(z_0)| < r\} \cap h(U) = \emptyset$ i.e. $|h(z) + h(z_0)| \geq r$ and $2|h(z_0)| \geq r$ for all $z \in U$. Now consider the injective holomorphic function

$$H(z) := \frac{r}{8}\cdot\frac{|h'(z_0)|}{|h(z_0)|^2}\cdot\frac{h(z_0)}{h'(z_0)}\cdot\frac{h(z)-h(z_0)}{h(z)+h(z_0)} \tag{40}$$

which has derivative

$$H'(z_0) = \frac{r}{16} \cdot \frac{|h'(z_0)|}{|h(z_0)|} > 0$$

Finally

$$\begin{aligned}
\left| \frac{h(z) - h(z_0)}{h(z) + h(z_0)} \right| &= |h(z_0)| \cdot \left| \frac{1}{h(z_0)} - \frac{2}{h(z) + h(z_0)} \right| \\
&\leq |h(z_0)| \left(\frac{1}{|h(z_0)|} + \frac{2}{|h(z) + h(z_0)} \right) \\
&\leq |h(z_0)| \left(\frac{2}{r} + \frac{2}{r} \right) = \frac{4|h(z_0)|}{r}
\end{aligned} \tag{41}$$

So $|H(z)| \leq \frac{1}{2} < 1$ for all $z \in U$ i.e. $H(U) \subset \mathbb{D}$.

Proof of claim 2:
Consider the sequence $\{f_n\}$ where $f_n \in \mathcal{F}$ for all $n \in \mathbb{N}$ and suppose the sequence converges to a function f with respect to the metric (23). Claim 3 is equivalent to the conjunction of the following subclaims:

Subclaim 3.1: f is holomorphic
Subclaim 3.2: f is either injective or identically zero
Subclaim 3.3: $f(z_0) = 0$
Subclaim 3.4: $f'(z_0) \geq 0$
Subclaim 3.5: $f(U) \subseteq \mathbb{D}$

$\mathcal{F}$ is clearly locally bounded: for any open disc $B \subset U$ then $|f(z)| < 1$ for all $z \in B$ for all $f \in \mathcal{F}$. By Montel's theorem the family $\mathcal{F}$ is satisfies the hypotheses for the Arzela-Ascoli Theorem. In the proof of Arzela-Ascoli, we noted that the limit f is necessarily uniformly continuous. Therefore by Proposition 3.24 f is holomorphic and we have proved subclaim 3.1.

Proposition 3.12 tells us that convergence with respect to the metric (23) is equivalent to uniform convergence on compact sets. The set $\{z_0\}$ is compact so this becomes pointwise convergence and proves subclaim 3.3.

Proposition 3.24 which tells us $f'_n \to f'$ uniformly on compact sets. Again we note that $\{z_0\}$ is a compact set so $f'_n(z_0) \to f_(z_0)$ which must therefore be non-negative as all $f'_n(z_0)$ are strictly greater than 0. This proves subclaim 3.4.

Clearly $f(U) \subseteq \overline{\mathbb{D}}$, the closed disc: if for some $z \in U$ we had $|f(z)| > 1$ then $f_n(z)$ could not converge to it. We appeal to the maximum modulus principle (Conway, 1995, 79) to claim that for all $z \in U$ we have $|f(z)| \neq 1$ because otherwise this would be a maximum modulus and f would be constant. This proves subclaim 3.5.

To prove subclaim 3.2, we shall use Theorem 4.2 (Hurwitz's Theorem) — specifically Corollary 4.3. Consider some point $z_1 \in U$ and let $\xi := f(z_1)$ and $\xi_n := f_n(z_1)$. Consider $z_2 \in U$ such that $z_1 \neq z_2$. We can find some $r > 0$ such that $z_1 \notin B(z_2; r)$. Let $h_n(z) := f_n(z) - \xi_n$ and observe that $h_n(z) \neq 0$ for all $z \in B(z_2; r)$. Also, $\{h_n\}_{n=1}^{\infty} \to h := f - \xi$. Apply Corollary 4.3 and conclude that h is either identically zero on $B(z_2; r)$ or nowhere zero on $B(z_2; r)$.

Suppose h is identically zero on $B(z_2; r)$. Holomorphic functions have isolated zeroes or are identically zero (a fact which can be readily proved using Corollary 2.20) and h is holomorphic so then h is identically zero on all of U and $f(z) = \xi = f(z_0) = 0$ for all $z \in U$.

Suppose h is nowhere zero on $B(z_2; r)$. Then $h(z_2) \neq 0$ so $f(z_2) \neq \xi = f(z_1)$. Since z_2 was arbitrary then f is injective.

<u>Proof of claim 3:</u>
We already noted in the proof of claim 2 that $\mathcal{F}$ is clearly locally bounded and by Montel's theorem the family $\mathcal{F}$ is normal. So the closure $\mathcal{F}^{-}$ is compact by theorem 3.16.

Note that the function $\Phi : f \to f'(z_0)$ is continuous (proposition 3.25). By claim 2, we know that Φ exists on $\mathcal{F}^{-}$ (all members are complex differentiable at z_0). The image of a compact set under a continuous function is also compact so $\Phi(\mathcal{F}^{-})$ is compact — note that by construction, and claim 2, $\Phi(\mathcal{F}^{-})$ is a subset of $\mathbb{R}$. Therefore $\Phi(\mathcal{F}^{-})$ is bounded and attains its bounds, so there exists a (not necessarily unique) function $f \in \mathcal{F}^{-}$ such that $f'(z_0) \geq g'(z_0)$ for all $g \in \mathcal{F}^{-} \supset \mathcal{F}$.

<u>Proof of claim 4:</u>
Let $g(U) \neq \mathbb{D}$. Then there is some $\omega_0 \in \mathbb{D}$ such that $g(z) \neq \omega_0$ for any $z \in U$. Since U is simply connected, then by Corollary 2.26 we know the following injective holomorphic function $G \colon U \to \mathbb{D}$ exists

$$G(z) = \sqrt{\frac{g(z) - \omega_0}{1 - \overline{\omega}_0 g(z)}} \tag{42}$$

We normalise the function as follows

$$h(z) = \frac{|G'(z_0)|}{G'(z_0)} \cdot \frac{G(z) - G(z_0)}{1 - \overline{G(z_0)}G(z)} \tag{43}$$

Clearly $h(z_0) = 0$. We find for the derivative

$$h'(z_0) = \frac{|G'(z_0)|}{1 - |G(z_0)|^2} = \frac{1 + |\omega_0|}{2\sqrt{|\omega_0|}} g'(z_0) > g'(z_0) \tag{44}$$

To convince the reader that $h'(z_0)$ really is greater than $g'(z_0)$ then observe

$$\left(\frac{1 + |\omega_0|}{2\sqrt{|\omega_0|}} \right)^2 = \frac{1 + 2|\omega_0| + |\omega_0|^2}{4|\omega_0|}$$

$$1 - 2|\omega_0| + |\omega_0|^2 = (1 - |\omega_0|)^2 > 0$$

$\square$

Remark 5.2. The Riemann Mapping Theorem tells us that any simply connected proper subdomain $U \subsetneq \mathbb{C}$ is conformally isomorphic to the open disc $\mathbb{D}$.

References

Ahlfors, Lars V. 1978. *Complex analysis*, 3rd ed., International series in pure and applied mathematics, McGraw-Hill Professional.

Conway, John B. 1995. *Functions of one complex variable I*, 2nd ed., Graduate Texts in Mathematics, Springer.

Munkres, James R. 1999. *Topology*, 2nd ed., Pearson.

Spivak, Michael. 2006. *Calculus*, 3rd ed., Cambridge University Press.

Stein, Elias M and Rami Shakarchi. 2003. *Complex analysis*, Princeton lectures in analysis, Princeton University Press.